ELEN E3401: Electromagnetics

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Lecture #6

COLUMBIA ENGINEERING
The Fu Foundation School of Engineering and Applied Science

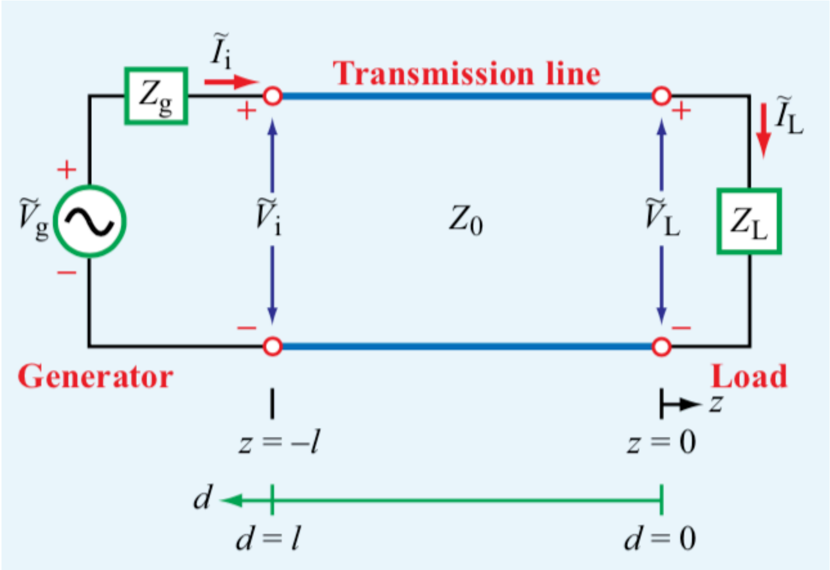
Lossless Transmission Line

Consider lossless line, $\gamma = j\beta$

Examine these wave solutions of wave equations:

$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0$$

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0$$



$$\tilde{V}(z) = V_0{}^{+}e^{-j\beta z} + V_0{}^{-}e^{j\beta z}$$

$$\tilde{I}(z) = \frac{V_0{}^{+}}{Z_0}e^{-j\beta z} - \frac{V_0{}^{-}}{Z_0}e^{j\beta z}$$

$$\Gamma = \frac{V_0{}^{-}}{V_0{}^{+}} \longrightarrow V_0{}^{-} = \Gamma\, V_0{}^{+}$$

$$\tilde{V}(z) = V_0{}^{+}(e^{-j\beta z} + \Gamma e^{j\beta z})$$

$$\tilde{I}(z) = \frac{V_0{}^{+}}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z})$$

Solutions have $+z$ and $-z$ wave components

Standing Waves

$$\tilde{V}(z) = {V_0}^+(e^{-j\beta z}+\Gamma e^{j\beta z})$$

Compute $|\tilde{V}(z)|$ → the magnitude of $\tilde{V}(z)$

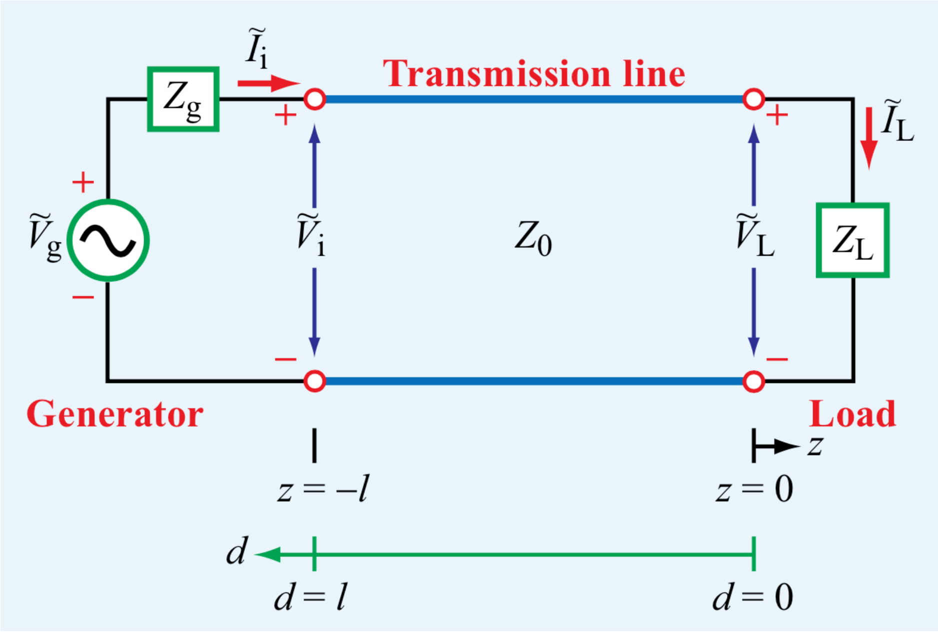
$$|\tilde{V}(z)| = |\tilde{V}(z)\tilde{V}(z)^*|^{\ 1/2}$$

$$\left|\tilde{V}(z)\right| = \left\{\left[{V_0}^+(e^{-j\beta z}+|\Gamma|e^{j\theta_r}e^{j\beta z})\ \right] \times \left[({V_0}^+)^*(e^{j\beta z}+|\Gamma|e^{-j\theta_r}e^{-j\beta z})\right]\right\}^{1/2}$$

$$= |{V_0}^+|\left[1 + |\Gamma|^2 + |\Gamma|(e^{j(2\beta z+\theta_r)} + e^{-j(2\beta z+\theta_r)})\right]^{1/2} \longleftarrow e^{jx} + e^{-jx} = 2\cos(x)$$

$$= |{V_0}^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

Standing Waves



Load at z = 0

Generator at z = -l

d = -z

Express $\tilde{V}$ as function of (d) instead of (z)   z → -d

$$\left|\tilde{V}(d)\right| = |V_0{}^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

We also obtain similar equations for $|\tilde{I}(d)|$

$$\left|\tilde{I}(d)\right| = \frac{|V_0{}^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

Standing Waves

$|\tilde{V}(d)|$ and $|\tilde{I}(d)|$ are standing waves

→ result of interference of two travelling waves components

Maximum value of standing wave occurs when incident and reflection components are in-phase:

$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

$|\tilde{V}(d)|$ max occurs when: $(2\beta d - \theta_r = 2n\pi)$

Consider example: $|V_0^+| = 1V$, $Z_0 = 50\ \Omega$, $\Gamma = 0.3e^{j30^o}$

$|\Gamma| = 0.3,\ \theta_r = 30°$

$$|\tilde{V}|_{max} = (1 + |\Gamma|)|V_0^+| = 1.3\text{V}$$

Standing Waves

Minimum value of $|\tilde{V}(d)|$ occurs when two waves, incident and reflective interfere destructively – opposite phase:

$$|\tilde{V}|_{min} \text{ occurs when: } (2\beta d - \theta_r = (2n+1)\pi)$$

$$|\tilde{V}|_{min} = (1 - |\Gamma|)\left|V_0^{\,+}\right| = 0.7\text{V}$$

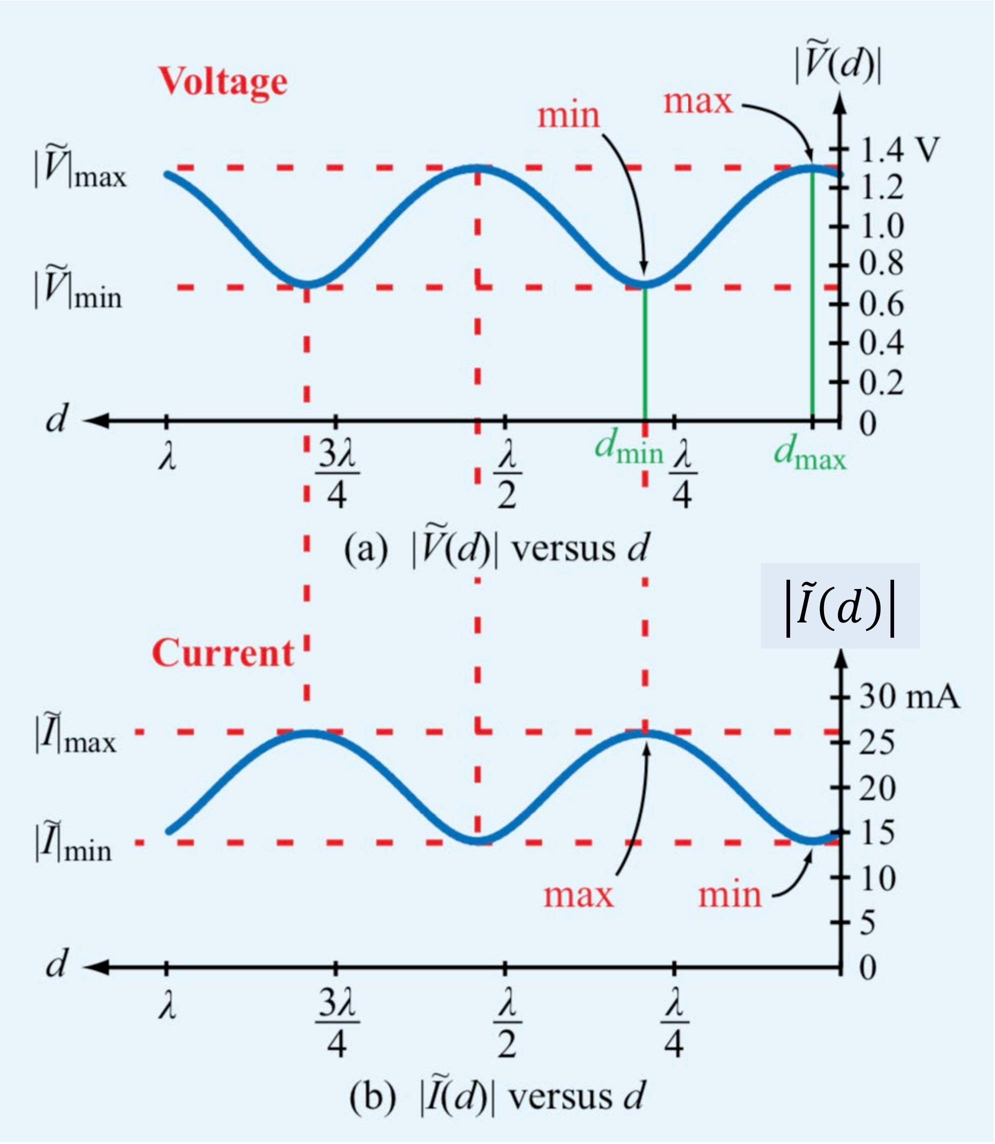
Note: spatial (λ) of standing wave = λ/2

$|\tilde{V}(d)|$ and $|\tilde{I}(d)|$ are opposite phase:

$$\left|\tilde{V}(d)\right| = |V_0^{\,+}|\left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)\right]^{1/2}$$

$$\left|\tilde{I}(d)\right| = \frac{|V_0^{\,+}|}{Z_0}\left[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)\right]^{1/2}$$

Standing Waves



Example of $|\tilde{V}(d)|$ and $|\tilde{I}(d)|$ shown for transmission line with:

$$|V_0^+| = 1\,V, \quad |\Gamma| = 0.3,$$
$$Z_0 = 50\ \Omega, \ \theta_r = 30°$$

$$Z_0 = 50\ \Omega \qquad \Gamma = 0.3e^{j30^o}$$

$$|\tilde{V}|_{max} = (1+|\Gamma|)\left|V_0^+\right| = 1.3\text{V}$$

$$|\tilde{V}|_{min} = (1-|\Gamma|)\left|V_0^+\right| = 0.7\text{V}$$

$$S = \frac{|\tilde{V}_{max}|}{|\tilde{V}_{min}|} = \frac{1.3}{0.7}$$

Voltage Standing Waves

Standing wave with maxima 2 $|{V_0}^+|$

$$\left|\tilde{V}(d)\right| = |{V_0}^+|[1 + |\Gamma|]$$

$$2\beta d_{max} - \theta_r = 2\pi n$$

$$d_{max} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r \lambda}{4\pi} + \frac{\lambda}{2} \qquad \beta = \frac{2\pi}{\lambda}$$

$$\begin{cases} n = 1,2, \dots if\, \theta_r < 0, \\ n = 0,1,2, \dots if\, \theta_r \geq 0. \end{cases} \qquad -\pi \leq \theta_r \leq \pi$$

If $\theta_r \geq 0$ - then first voltage maximum, at n = 0 occurs at:

$$d_{max} = \frac{\theta_r \lambda}{4\pi}$$

If $\theta_r < 0$ - then first voltage maximum, at n = 1 occurs at:

$$d_{max} = \frac{\theta_r \lambda}{4\pi} + \frac{\lambda}{2}$$

Voltage Standing Waves

$|\tilde{V}|_{min}$ → will occur when cos argument = (2n+1) π

$$|\tilde{V}|_{min} = \left|V_0^{\ +}\right|(1 - |\Gamma|)$$

when $2\beta d_{min} - \theta_r = (2n + 1)\pi$

$-\pi \le \theta_r \le \pi$ → then first minimum occurs at n = 0, or $\frac{\lambda}{4}$ from d_{max}

$$d_{min} = \begin{cases} d_{max} + \dfrac{\lambda}{4}, & if\, d_{max} < \dfrac{\lambda}{4} \\ d_{max} - \dfrac{\lambda}{4}, & if\, d_{max} \ge \dfrac{\lambda}{4} \end{cases}$$

Voltage Standing Wave Ratio

(Dimensionless) $$S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

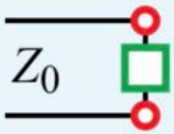
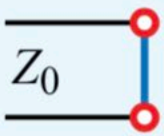
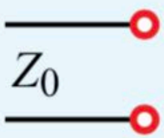
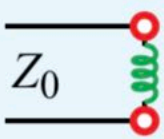
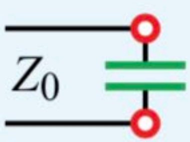
VSWR or SWR

This is a measure of load mismatch versus matched load.

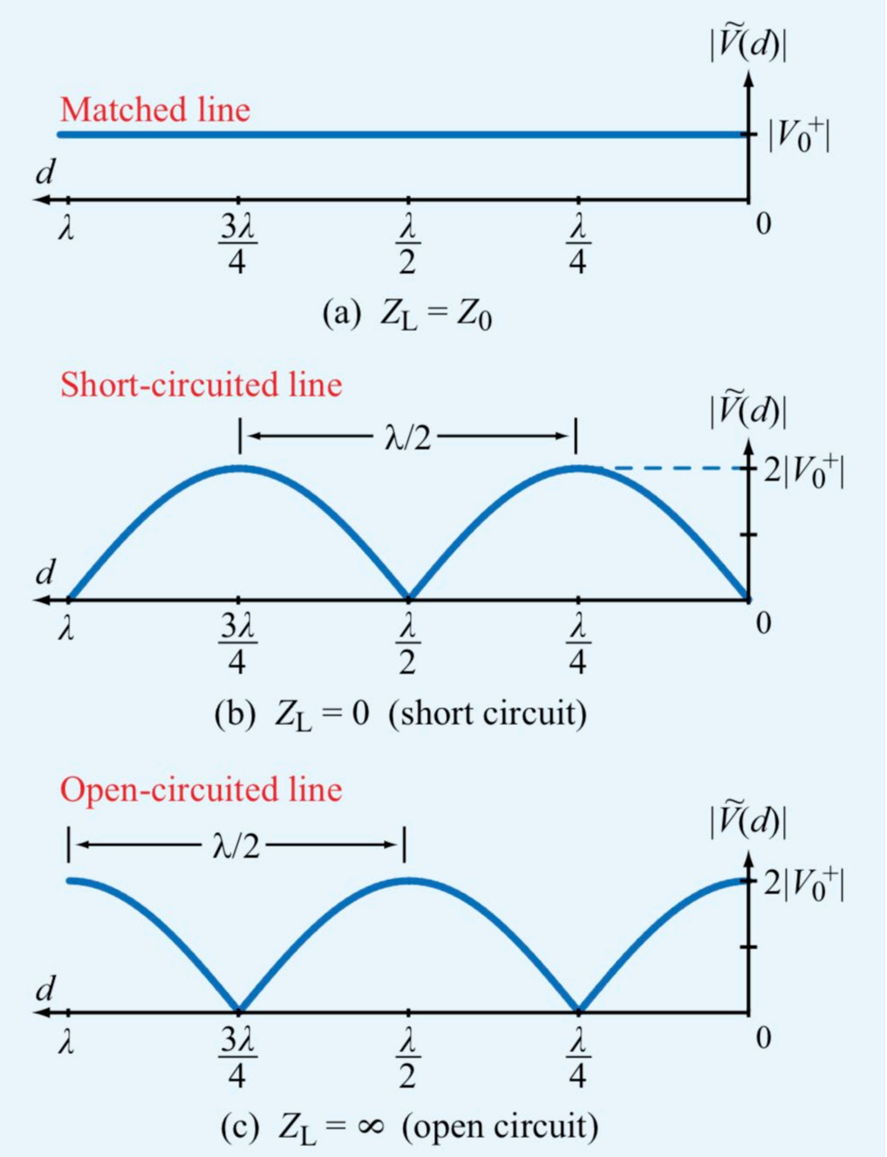
Matched load: $\Gamma = 0 \rightarrow S = 1$

For $|\Gamma| = 1 \rightarrow S = \infty$

Reflection Coefficient: $\Gamma = |\Gamma| e^{j\theta_r}$

Load	$\lvert\Gamma\rvert$	θ_r
 Z_0 (Matched Load)	0 (no reflection)	irrelevant
 (short)	1	$\pm 180°$ (phase opposition)
 (open)	1	0 (in-phase)
 $jX = j\omega L$ (Inductive Load)	1	$\pm 180° - 2\tan^{-1} x$
 $jX = \dfrac{-j}{\omega C}$ (Capacitive Load)	1	$\pm 180° + 2\tan^{-1} x$

Voltage Standing Waves



1. Matched line $Z_L = Z_0$, $|\Gamma|=0$

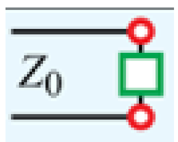
   $|\tilde{V}(d)| = |V_0^{+}|$ for all d

2. Short circuit $Z_L = 0$, $|\Gamma|=1$, $\Gamma = -1$

3. Open circuit $Z_L = \infty$, $|\Gamma|=1$, $\Gamma = 1$

Reflection Coefficient $\Gamma = |\Gamma| e^{j\theta_r}$

Load - general $z_L = \frac{Z_L}{Z_0} = \frac{R + jX}{Z_0} = r + jx$



$Z_L = (r + jx)Z_o$ $|\Gamma| = \left[\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$

$r = \frac{R}{Z_0}$ $x = \frac{X}{Z_0}$ of z_L $\theta_r = tan^{-1}\left(\frac{x}{r-1}\right) - tan^{-1}\left(\frac{x}{r+1}\right)$

Real Imaginary

Example: purely reactive load

<u>Q: Show |*Γ*| = 1 for lossless line connected to a purely reactive load</u>

$$Z_L = jX_L \qquad \Gamma = \frac{{V_0}^-}{{V_0}^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

$$\Gamma = \frac{-(Z_0 - jX_L)}{Z_0 + jX_L} = \frac{-\sqrt{{Z_0}^2 + {X_L}^2}e^{-j\theta}}{\sqrt{{Z_0}^2 + {X_L}^2}e^{j\theta}}$$

Note: $\theta = \tan^{-1}(\frac{X_L}{Z_0})$

$$\Gamma = -e^{-j2\theta}$$

$$|\,\Gamma\,| = \sqrt[+]{e^{-j2\theta}e^{j2\theta}}$$

Note: $|z| = \sqrt[+]{zz^*}$

$$|\,\Gamma\,| = 1$$